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On a generalization of distance sets

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Abstract

A subset X in the d-dimensional Euclidean space is called a k-distance set if there are exactly k distinct distances between two distinct points in X and a subset X is called a locally k-distance set if for any point x in X, there are at most k distinct distances between x and other points in X.

Delsarte, Goethals, and Seidel gave the Fisher type upper bound for the cardinalities of k-distance sets on a sphere in 1977. In the same way, we are able to give the same bound for locally k-distance sets on a sphere. In the first part of this paper, we prove that if X is a locally k-distance set attaining the Fisher type upper bound, then determining a weight function w, (X, w) is a tight weighted spherical $2k$-design. This result implies that locally k-distance sets attaining the Fisher type upper bound are k-distance sets. In the second part, we give a new absolute bound for the cardinalities of k-distance sets on a sphere. This upper bound is useful for k-distance sets for which the linear programming bound is not applicable. In the third part, we discuss about locally two-distance sets in Euclidean spaces. We give an upper bound for the cardinalities of locally two-distance sets in Euclidean spaces. Moreover, we prove that the existence of a spherical two-distance set in $(d-1)$-space which attains the Fisher type upper bound is equivalent to the existence of a locally two-distance set but not a two-distance set in d-space with more than $d(d+1)/2$ points. We also classify optimal (largest possible) locally two-distance sets for dimensions less than eight. In addition, we determine the maximum cardinalities of locally two-distance sets on a sphere for dimensions less than forty.

1 Introduction

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. For $X \subset \mathbb{R}^d$, let $A(X) = \{d(x,y) | x, y \in X, x \neq y\}$ where $d(x,y)$ is the Euclidean distance between x and y in $\mathbb{R}^d$. We call X a *k-distance set* if $|A(X)| = k$. Moreover for any $x \in X$, define $A_X(x) = \{d(x,y) | y \in X, x \neq y\}$. We will abbreviate $A(x) = A_X(x)$ whenever there is no risk of confusion. A subset $X \subset \mathbb{R}^d$ is called a *locally k-distance set* if $|A_X(x)| \leq k$ for all $x \in X$. Clearly every k-distance set is a locally k-distance set. A locally k-distance set is said to be *proper* if it is not a k-distance set. Two subsets in $\mathbb{R}^d$ are said to be isomorphic if there exists a similar transformation from one to the other. An interesting problem for k-distance sets (resp. locally k-distance set) is to determine the largest possible cardinality of k-distance sets (resp. locally k-distance set) in $\mathbb{R}^d$. We denote this number by $DS_d(k)$ (resp. $LDS_d(k)$) and a k-distance set X (resp. locally k-distance set X) in $\mathbb{R}^d$ is said to be *optimal* if $|X| = DS_d(k)$ (resp. $LDS_d(k)$). Moreover we denote the maximum cardinality of a k-distance set (resp. locally k-distance set) in the unit sphere $S^{d-1} \subset \mathbb{R}^d$ by $DS^*_d(k)$ (resp. $LDS^*_d(k)$).

For upper bounds on the cardinalities of distance sets in $\mathbb{R}^d$, Bannai-Bannai-Stanton [4] and Blokhuis [8] gave $DS_d(k) \leq \binom{d+k}{k}$. For $k = 2$, the numbers $DS_d(2)$ are known for $d \leq 8$ (Kelly [18], Croft [9]

and Lisoněk [20]). For $d = 2$, the numbers $DS_2(k)$ are known and optimal k-distance sets are classified for $k \leq 5$ (Erdős-Fishburn [15], Shinohara [22], [23]). Moreover we have $DS_3(3) = 12$ and every optimal three-distance set is isomorphic to the set of vertices of a regular icosahedron (Shinohara [24]).

d	1	2	3	4	5	6	7	8
$DS_d(2)$	3	5	6	10	16	27	29	45

k	1	2	3	4	5
$DS_2(k)$	3	5	7	9	12

Table: Maximum cardinalities for two-distance sets and planar k-distance sets

We have a lower bound for $DS_d^*(2)$ of $d(d+1)/2$ since the set of all midpoints of the edges of a d-dimensional regular simplex is a two-distance set on a sphere with $d(d+1)/2$ points. Musin determined that $DS_d^*(2) = d(d+1)/2$ for $7 \leq d \leq 21$, $24 \leq d \leq 39$ [21]. For $2 \leq d \leq 6$, we have $DS_d^*(2) = DS_d(2)$ and for $d = 22$, we have $DS_d^*(2) = 275$. For $d = 23$, $DS_d^*(2) = 276$ or 277 [21].

Delsarte, Goethals, and Seidel gave the Fisher type upper bound for the cardinalities of k-distance sets on a sphere [11]. This upper bound also applies to locally k-distance sets on a sphere.

Theorem 1.1 (Fisher type inequality [11])**.** (i) *Let X be a locally k-distance set on S^{d-1}. Then, $|X| \leq \binom{d+k-1}{k} + \binom{d+k-2}{k-1} (=: N_d(k))$.*
(ii) *Let X be an antipodal (i.e. for any $x \in X$, $-x \in X$) locally k-distance set on S^{d-1}. Then, $|X| \leq 2\binom{d+k-2}{k-1} (=: N'_d(k))$.*

It is well known that if a k-distance set X attains this upper bound, then X is a tight spherical design. We will give the definition of spherical designs in the next section. Of course, k-distance sets which attain this upper bound are optimal. This optimal k-distance set is very interesting because of its relationship with the design theory. Classification of tight spherical t-designs have been well studied in [5, 6, 7]. Classifications of tight spherical t-designs are complete, except for $t = 4, 5, 7$. This implies that classifications of k-distance sets (resp. antipodal k-distance sets) which attain this upper bound are complete, except for $k = 2$ (resp. $k = 3, 4$). For $t = 4$, a tight spherical four-design in S^{d-1} exists only if $d = 2$ or $d = (2l+1)^2 - 3$ for a positive integer l and the existence of a tight spherical four-design in S^{d-1} is known only for $d = 2, 6$ or 22.

In Section 2, we prove the following theorem.

Theorem 1.2. (i) *Let X be a locally k-distance set on S^{d-1}. If $|X| = N_d(k)$, then for some determined weight function w, (X, w) is a tight weighted spherical $2k$-design. Conversely, if (X, w) is a tight weighted spherical $2k$-design, then X is a locally k-distance set (indeed, X is a k-distance set).*
(ii) *Let X be an antipodal locally k-distance set on S^{d-1}. If $|X| = N'_d(k)$, then for some determined weight function w, (X, w) is a tight weighted spherical $(2k-1)$-design. Conversely, if (X, w) is a tight weighted spherical $(2k-1)$-design, then X is an antipodal locally k-distance set (indeed, X is an antipodal k-distance set).*

This theorem implies that the concept of locally distance sets is a natural generalization of distance sets, because this theorem is a generalization of the relationship between tight spherical designs and distance sets.

Indeed, Theorem 1.2 implies the following.

Theorem 1.3. (i) *Let X be a locally k-distance set on S^{d-1}. If $|X| = N_d(k)$, then X is a k -distance set.*
(ii) *Let X be an antipodal locally k-distance set on S^{d-1}. If $|X| = N'_d(k)$, then X is a k-distance set.*

In Section 3, we give a new upper bound for k-distance sets on S^{d-1}. This upper bound is useful for k-distance sets to which the linear programming bound is not applicable.

In Section 4, we discuss locally two-distance sets in $\mathbb{R}^d$. We first give an upper bound for the cardinalities of locally two-distance sets. Moreover, we mention that every proper locally two-distance set in $\mathbb{R}^d$ with more than $d(d+1)/2$ points contains a two-distance set in S^{d-2} which attains the Fisher type upper bound. Note that a two-distance set in $\mathbb{R}^d$ with $d(d+1)/2$ points exists. We also classify optimal locally two-distance sets in $\mathbb{R}^d$ for $d < 8$. In addition, we determine $LDS_2^*(d)$ for $d < 40$ by using the value of $DS_d^*(2)$ for $d < 40$. In particular, we do not know $DS_{23}^*(2)$ but can determine $LDS_{23}^*(2)$.

2 Locally distance sets and weighted spherical designs

We prove Theorem 1.2 in this section. First, we give the definition of weighted spherical designs.

Definition 2.1 (Weighted spherical designs)**.** Let X be a finite set on S^{d-1}. Let w be a weight function: $w: X \to \mathbb{R}_{>0}$, such that $\sum_{x \in X} w(x) = 1$. (X, w) is called a weighted spherical t-design if the following equality holds for any polynomial f in d variables and of degree at most t:

$$\frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) d\sigma(x) = \sum_{x \in X} w(x) f(x),$$

where the left hand side involves the integral of f on the sphere. X is called a spherical t-design if $w(x) = 1/|X|$ for all $x \in X$.

We have the following lower bound for the cardinalities of weighted spherical t-designs.

Theorem 2.2 (Fisher type inequality [11, 12])**.** (i) *Let X be a weighted spherical $2e$-design. Then, $|X| \geq \binom{d+e-1}{e} + \binom{d+e-2}{e-1} = N_d(e)$.*
(ii) *Let X be a weighted spherical $(2e-1)$-design. Then, $|X| \geq 2\binom{d+e-2}{e-1} = N'_d(e)$.*

If equality holds, X is said to be tight. The following theorem shows a strong relationship between tight spherical t-designs and k-distance sets.

Theorem 2.3 (Delsarte, Goethals and Seidel [11])**.** (i) *X is a k-distance set on S^{d-1} with $N_d(k)$ points if and only if X is a tight spherical $2k$-design.*
(ii) *X is an antipodal k-distance set on S^{d-1} with $N'_d(k)$ points if and only if X is a tight spherical $(2k-1)$-design.*

Remark 2.4. In particular, X is a two-distance set on S^{d-1} with $N_d(2)$ points if and only if X is a tight spherical four-design. X is an antipodal three-distance set on S^{d-1} with $N'_d(2)$ points if and only if X is a tight spherical five-design. Note that the existence of a tight spherical four-design on S^{d-2} is equivalent to the existence of a tight spherical five-design on S^{d-1}. Let X be a tight spherical five-design on S^{d-1}. Then, we can put $A(X) = \{\alpha, \beta, 2\}$ $(\alpha < \beta)$. For a fixed $x \in X$, we define $X_\alpha := \{y \in X \mid d(x, y) = \alpha\}$. Then, we can regard X_α as a tight spherical four-design on S^{d-2}. This relationship between tight four-designs and five-designs is important in Section 4.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set on S^{d-1}. Let $\mathrm{Harm}_l(\mathbb{R}^d)$ be the linear space of all real harmonic homogeneous polynomials of degree l, in d variables. We put $h_l := \dim(\mathrm{Harm}_l(\mathbb{R}^d))$. Let $\{\varphi_{l,i}\}_{i=0,1,\ldots,h_l}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^{d-1})$ with respect to the inner product $\langle f, g\rangle = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)g(x)d\sigma(x)$. Let H_l be the characteristic matrix of degree l, that is, its (i, j)-th entry is $\varphi_{l,j}(x_i)$. The following gives the definition of Gegenbauer polynomials and discusses the Addition Formula which will be used in the succeeding discussion.

Definition 2.5. Gegenbauer polynomials are a set of orthogonal polynomials $\{G_l^{(d)}(t) \mid l = 1, 2, \ldots\}$ of one variable t. For each l, $G_l^{(d)}(t)$ is a polynomial of degree l, defined in the following manner.

1. $G_0^{(d)}(t) \equiv 1$, $G_1^{(d)}(t) = dt$.
2. $tG_l^{(d)}(t) = \lambda_{l+1} G_{l+1}^{(d)}(t) + (1 - \lambda_{l-1}) G_{l-1}^{(d)}(t)$ for $l \geq 1$, where $\lambda_l = \frac{l}{d+2l-2}$.

Note that $G_l^{(d)}(1) = \dim(\mathrm{Harm}_l(\mathbb{R}^d)) = h_l$. Let $(,)$ be the standard inner product in $\mathbb{R}^d$.

Theorem 2.6 (Addition formula [11, 1])**.** *For any x, y on S^{d-1}, we have*

$$\sum_{k=1}^{h_l} \varphi_{l,k}(x)\varphi_{l,k}(y) = G_l^{(d)}((x, y)).$$

Let I be the identity matrix, and ${}^tN$ be the transpose of a matrix N. The following is a key theorem to prove Theorem 1.3.

Theorem 2.7. *The following are equivalent:*
(i) (X, w) *is a weighted spherical t-design.*
(ii) ${}^tH_eWH_e = I$ *and* ${}^tH_eWH_r = 0$ *for* $e = \lfloor \frac{t}{2} \rfloor$ *and* $r = e-(-1)^t$*. Here,* $W = \mathrm{Diag}\{w(x_1), w(x_2), \ldots, w(x_n)\}$*.*

We require the two following lemmas in order to prove Theorem 2.7.

Lemma 2.8 (Lemma 3.2.8 in [1] or [11])**.** *We have the Gegenbauer expansion* $G_k^{(d)}G_l^{(d)} = \sum_{i=0}^{k+l} q_i(k,l)G_i^{(d)}$*. Then, the following hold.*
(i) *For any* i, k *and* l*,* $q_i(k,l) \geq 0$*.*
(ii) *For any* k *and* l*,* $q_0(k,l) = h_k\delta_{k,l}$*, where* $\delta_{k,l} = 1$ *if* $k = l$ *and* $\delta_{k,l} = 0$ *if* $k \neq l$*.*
(iii) $q_i(k,l) \neq 0$ *if and only if* $|k-l| \leq i \leq k+l$ *and* $i \equiv k+l \mod 2$*.*

For an $m \times n$ matrix M, we define $||M||^2 := \sum_{i=1}^m \sum_{j=1}^n M(i,j)^2$, namely the sum of squares of all matrix entries.

Lemma 2.9. *For* $k + l \geq 1$*,*

$$\left|\left|{}^tH_kWH_l - \Delta_{k,l}\right|\right|^2 = \sum_{i=1}^{k+l} q_i(k,l) \left|\left|{}^tH_iWH_0\right|\right|^2 \tag{1}$$

where

$$\Delta_{k,l} = \begin{cases} I, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases}.$$

Proof. Note that

$$||{}^tH_kWH_l||^2 = \sum_{i=1}^{h_k}\sum_{j=1}^{h_l}\left(\sum_{x\in X} w(x)\varphi_{k,i}(x)\varphi_{l,j}(x)\right)^2 \tag{2}$$

$$= \sum_{x\in X}\sum_{y\in X} w(x)w(y)\sum_{i=1}^{h_k}\varphi_{k,i}(x)\varphi_{k,i}(y)\sum_{j=1}^{h_l}\varphi_{l,j}(x)\varphi_{l,j}(y) \tag{3}$$

$$= \sum_{x\in X}\sum_{y\in X} w(x)w(y)G_k^{(d)}((x,y))G_l^{(d)}((x,y)).$$

When $l = 0$, we have

$$||{}^tH_kWH_0||^2 = \sum_{x\in X}\sum_{y\in X} w(x)w(y)G_k^{(d)}((x,y)). \tag{4}$$

If $k \neq l$, then

$$\begin{aligned}
||{}^tH_kWH_l||^2 &= \sum_{x\in X}\sum_{y\in X} w(x)w(y)G_k^{(d)}((x,y))G_l^{(d)}((x,y)) \\
&= \sum_{x\in X}\sum_{y\in X} w(x)w(y)\sum_{i=0}^{k+l} q_i(k,l)G_i^{(d)}((x,y)) \\
&= \sum_{i=0}^{k+l} q_i(k,l)||{}^tH_iWH_0||^2 \qquad (\because \text{equality } (4)) \\
&= \sum_{i=1}^{k+l} q_i(k,l)||{}^tH_iWH_0||^2 \qquad (\because \text{Lemma 2.8}).
\end{aligned}$$

If $k = l$, then the summation of the squares of the diagonal entries is

$$\begin{aligned}
\sum_{i=1}^{h_k}\Big(({}^tH_kWH_k - I)(i,i)\Big)^2 &= \sum_{i=1}^{h_k}\Big(\sum_{x\in X} w(x)\varphi_{k,i}(x)\varphi_{k,i}(x) - 1\Big)^2 \\
&= \sum_{i=1}^{h_k}\Bigg(\Big(\sum_{x\in X} w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\Big)^2 - 2\sum_{x\in X} w(x)\varphi_{k,i}(x)\varphi_{k,i}(x) + 1\Bigg) \\
&= \sum_{i=1}^{h_k}\Big(\sum_{x\in X} w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\Big)^2 - 2\sum_{x\in X} w(x)\sum_{i=1}^{h_k}\varphi_{k,i}(x)\varphi_{k,i}(x) + h_k \\
&= \sum_{i=1}^{h_k}\Big(\sum_{x\in X} w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\Big)^2 - 2\sum_{x\in X} w(x)G_k^{(d)}(1) + h_k \\
&= \sum_{i=1}^{h_k}\Big(\sum_{x\in X} w(x)\varphi_{k,i}(x)\varphi_{k,i}(x)\Big)^2 - h_k
\end{aligned}$$

Therefore,

$$\begin{aligned}
\big|\big|{}^tH_kWH_k - I\big|\big|^2 &= ||{}^tH_kWH_k||^2 - h_k \\
&= \sum_{i=0}^{2k} q_i(k,k)||{}^tH_iWH_0||^2 - h_k \\
&= \sum_{i=1}^{2k} q_i(k,k)||{}^tH_iWH_0||^2. \qquad (5)
\end{aligned}$$

□

Proof of Theorem 2.7. (i) ⇒ (ii) is clear. We prove (ii) ⇒ (i). By Lemma 2.9,

$$\big|\big|{}^tH_eWH_e - I\big|\big|^2 = \sum_{i=1}^{2e} q_i(e,e)\,\big|\big|{}^tH_iWH_0\big|\big|^2 = 0. \qquad (6)$$

We have ${}^tH_iWH_0 = 0$ for even $i \le t$, because $q_i(e,e) > 0$ for even i, and $q_i(e,e) = 0$ for odd i. On the other hand,

$$\big|\big|{}^tH_eWH_r\big|\big|^2 = \sum_{i=1}^{2e-(-1)^t} q_i(e,r)\,\big|\big|{}^tH_iWH_0\big|\big|^2 = 0. \qquad (7)$$

We have ${}^tH_iWH_0 = 0$ for odd $i \le t$, because $q_i(e,r) > 0$ for odd i, and $q_i(e,r) = 0$ for even i. Therefore, these imply that for any $f \in \mathrm{P}_t(S^{d-1})$, the following equality holds:

$$\frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)d\sigma(x) = \sum_{x\in X} w(x)f(x).$$

□

Proof of Theorem 1.2. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a locally k-distance set on S^{d-1}. Suppose $|X| = N_d(k)$. For each $x \in X$, we define $A_{\mathrm{inn}}(x) := \{(x,y) \mid y \in X, x \neq y\}$. For each $x \in X$, we define the polynomial in d variables:

$$F_x(\xi) := (x,\xi)^{k-|A_{\mathrm{inn}}(x)|}\prod_{\alpha\in A_{\mathrm{inn}}(x)}\frac{(x,\xi)-\alpha}{1-\alpha},$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$. $F_x(\xi)$ is of degree k for all $x \in X$. For all $x_i, x_j \in X$, $F_{x_i}(x_j) = \delta_{i,j}$. We have the Gegenbauer expansion:

$$F_x(\xi) = \sum_{i=0}^{k} f_i^{(x)} G_i^{(d)}((x, \xi))$$

where $f_i^{(x)}$ are real numbers. In particular, we remark that $f_k^{(x)} > 0$ for every $x \in X$. By the addition formula,

$$F_x(\xi) = \sum_{i=0}^{k} f_i^{(x)} G_i^{(d)}((x, \xi)) = \sum_{i=0}^{k} f_i^{(x)} \sum_{j=1}^{h_i} \varphi_{i,j}(x)\varphi_{i,j}(\xi) \tag{8}$$

for $\xi \in S^{d-1}$. We define the diagonal matrices $C_i := \mathrm{Diag}\{f_i^{(x_1)}, f_i^{(x_2)}, \ldots, f_i^{(x_n)}\}$ for $0 \leq i \leq k$. $[C_0H_0, C_1H_1, \ldots, C_kH_k]$ and $[H_0, H_1, \ldots H_k]$ are $n \times n$ matrices. By the equality (8), we have the equality:

$$[C_0H_0, C_1H_1, \ldots, C_kH_k] \begin{bmatrix} {}^tH_0 \\ {}^tH_1 \\ \vdots \\ {}^tH_k \end{bmatrix} = [F_{x_i}(x_j)]_{i,j} = I. \tag{9}$$

Therefore, $[C_0H_0, C_1H_1, \ldots, C_kH_k]$ and $[H_0, H_1, \ldots H_k]$ are non-singular matrices. Thus,

$$\begin{bmatrix} {}^tH_0 \\ {}^tH_1 \\ \vdots \\ {}^tH_k \end{bmatrix} [C_0H_0, C_1H_1, \ldots, C_kH_k] = I \tag{10}$$

$$\begin{bmatrix} {}^tH_0C_0H_0 & {}^tH_0C_1H_1 & \cdots & {}^tH_0C_kH_k \\ {}^tH_1C_0H_0 & {}^tH_1C_1H_1 & \cdots & {}^tH_1C_kH_k \\ \vdots & \vdots & \ddots & \vdots \\ {}^tH_kC_0H_0 & {}^tH_kC_1H_1 & \cdots & {}^tH_kC_kH_k \end{bmatrix} = I. \tag{11}$$

Therefore, ${}^tH_kC_kH_k = I$ and ${}^tH_{k-1}C_kH_k = 0$. If we define the weight function $w(x) := f_k^{(x)}$ for $x \in X$, then X is a tight weighted spherical $2k$-design on S^{d-1} by Theorem 2.7.

Antipodal case Let X be an antipodal k-distance set with $N'_d(k)$ on S^{d-1}. There exist a subset Y such that $X = Y \cup (-Y)$ and $|X| = 2|Y|$. We define $A^2_{\mathrm{inn}}(x) := \{(x, y)^2 \mid y \in X, y \neq \pm x\}$ and

$$\varepsilon = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

For each $y \in Y$, we define the polynomial in d variables

$$F_y(\xi) := (y, \xi)^{k-1-2|A^2_{\mathrm{inn}}(y)\setminus\{0\}|} \prod_{0 \neq \alpha^2 \in A^2_{\mathrm{inn}}(y)} \frac{(y, \xi)^2 - \alpha^2}{1 - \alpha^2}.$$

$F_y(\xi)$ is of degree $k-1$ for all $y \in Y$. For all $y_i, y_j \in Y$, $F_{y_i}(y_j) = \delta_{i,j}$. We have the Gegenbauer expansion:

$$F_y(\xi) = \sum_{i=0}^{k-1} f_i^{(y)} G_i^{(d)}((y, \xi)).$$

Note that $f_i = 0$ for $i \equiv k \mod 2$. In particular, we remark that $f_{k-1}^{(y)} > 0$ for every $y \in Y$. We define the diagonal matrices $C_i := \mathrm{Diag}\{f_i^{(y_1)}, f_i^{(y_2)}, \ldots, f_i^{(y_{n/2})}\}$ for $0 \leq i \leq k-1$. Let $H_l^{(Y)}$ be the characteristic

matrix with respect to Y. $[C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2}H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1}H_{k-1}^{(Y)}]$ and $[H_\varepsilon^{(Y)}, H_{\varepsilon+2}^{(Y)}, \dots, H_{k-1}^{(Y)}]$ are $n/2 \times n/2$ matrices. By the addition formula, we have the equality:

$$[C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2}H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1}H_{k-1}^{(Y)}] \begin{bmatrix} {}^tH_\varepsilon^{(Y)} \\ {}^tH_{\varepsilon+2}^{(Y)} \\ \vdots \\ {}^tH_{k-1}^{(Y)} \end{bmatrix} = I. \tag{12}$$

Therefore, $[C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2}H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1}H_{k-1}^{(Y)}]$ and $[H_\varepsilon^{(Y)}, H_{\varepsilon+2}^{(Y)}, \dots, H_{k-1}^{(Y)}]$ are non-singular matrices. Thus,

$$\begin{bmatrix} {}^tH_\varepsilon^{(Y)} \\ {}^tH_{\varepsilon+2}^{(Y)} \\ \vdots \\ {}^tH_{k-1}^{(Y)} \end{bmatrix} [C_\varepsilon H_\varepsilon^{(Y)}, C_{\varepsilon+2}H_{\varepsilon+2}^{(Y)}, \dots, C_{k-1}H_{k-1}^{(Y)}] = I \tag{13}$$

$$\begin{bmatrix} {}^tH_\varepsilon^{(Y)}C_\varepsilon H_\varepsilon^{(Y)} & {}^tH_\varepsilon^{(Y)}C_{\varepsilon+2}H_{\varepsilon+2}^{(Y)} & \cdots & {}^tH_\varepsilon^{(Y)}C_{k-1}H_{k-1}^{(Y)} \\ {}^tH_{\varepsilon+2}^{(Y)}C_\varepsilon H_\varepsilon^{(Y)} & {}^tH_{\varepsilon+2}^{(Y)}C_{\varepsilon+2}H_{\varepsilon+2}^{(Y)} & \cdots & {}^tH_{\varepsilon+2}^{(Y)}C_{k-1}H_{k-1}^{(Y)} \\ \vdots & \vdots & \ddots & \vdots \\ {}^tH_{k-1}^{(Y)}C_\varepsilon H_\varepsilon^{(Y)} & {}^tH_{k-1}^{(Y)}C_{\varepsilon+2}H_{\varepsilon+2}^{(Y)} & \cdots & {}^tH_{k-1}^{(Y)}C_{k-1}H_{k-1}^{(Y)} \end{bmatrix} = I. \tag{14}$$

Therefore, ${}^tH_{k-1}^{(Y)}C_{k-1}H_{k-1}^{(Y)} = I$. Let H_l be a characteristic matrix with respect to X. We select the weight function $w(x) := f_{k-1}^{(x)}/2$ and $w(-x) = w(x)$ for $x \in X$. Since X is antipodal, this implies ${}^tH_{k-1}WH_{k-1} = I$ and ${}^tH_{k-1}WH_k = 0$. Therefore, X is a tight weighted spherical $(2k-1)$-design by Theorem 2.7.

($\Leftarrow$) It is known that tight weighted spherical $2k$-designs (resp. $(2k-1)$-design) are tight spherical $2k$-design (resp. $(2k-1)$-design) [25, 2, 3]. Therefore, a tight weighted spherical $2k$-design (resp. $(2k-1)$-design) is a k-distance set (resp. antipodal k-distance set). $\square$

Theorem 1.2 implies that (resp. antipodal) locally k-distance sets attaining the Fisher type upper bound are (resp. antipodal) k-distance sets .

3 A new upper bound for k-distance sets on S^{d-1}

The following upper bound for the cardinalities of k-distance sets is well known.

Theorem 3.1 (Linear programming bound [11]). *Let X be a k-distance set on S^{d-1}. We define the polynomial $F_X(t) := \prod_{\alpha \in A_{\mathrm{inn}}(X)}(t-\alpha)$ for X where $A_{\mathrm{inn}}(X) := \{(x,y) \mid x,y \in X, x \neq y\}$. We have the Gegenbauer expansion*

$$F_X(t) = \prod_{\alpha \in A_{\mathrm{inn}}(X)}(t-\alpha) = \sum_{i=0}^{k} f_i G_i^{(d)}(t),$$

where f_i are real numbers. If $f_0 > 0$ and $f_i \geq 0$ for all $1 \leq i \leq k$, then

$$|X| \leq \frac{F_X(1)}{f_0}.$$

This upper bound is very useful when $A_{\mathrm{inn}}(X)$ is given. However, if some f_i happens to be negative, then we have no useful upper bound for the cardinalities of k-distance sets. In this section, we give a useful upper bound for this case. A proof of the following theorem builds upon Delsarte's ideas for the binary codes [10].

Theorem 3.2. *Let X be a k-distance set on S^{d-1}. We define the polynomial $F_X(t)$ of degree k:*

$$F_X(t) := \prod_{\alpha \in A_{\mathrm{inn}}(X)} (t-\alpha) = \sum_{i=0}^{k} f_i G_i^{(d)}(t),$$

where f_i are real numbers. Then,

$$|X| \leq \sum_{i \text{ with } f_i>0} h_i, \tag{15}$$

where the summation is over i with $0 \leq i \leq k$ satisfying $f_i > 0$ and $h_i = \dim(\mathrm{Harm}_i(\mathbb{R}^d))$.

Proof. Let $X := \{x_1, x_2, \dots, x_n\}$ be a k-distance set on S^{d-1}. Let $\{\varphi_{l,k}\}_{1 \leq k \leq h_l}$ be an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^d)$. H_l is the characteristic matrix. We have the Gegenbauer expansion $F_X(t) = \prod_{\alpha \in A_{\mathrm{inn}}(X)} \frac{t-\alpha}{1-\alpha} = \sum_{i=0}^{k} f_i G_i^{(d)}(t)$. Define the $\sum_{i=0}^{k} h_i \times n$ matrix $H := {}^t[H_0, H_1, \dots, H_k]$. By the addition formula, we get

$${}^tHFH = I_n$$

where I_m is the identity matrix of degree m, and $F = f_0 I_1 \oplus f_1 I_{h_1} \oplus \cdots \oplus f_s I_{h_s}$ (direct sum). Therefore, the column vectors of H are linearly independent, and lie in the positive subspace of the quadratic form F. Thus, n can not exceed the number of the positive entries of F. $\square$

If $f_i > 0$ for all $0 \leq i \leq k$, then this upper bound is the same as the Fisher type inequality.

By using a similar method, we prove a similar upper bound for the antipodal case.

Theorem 3.3 (Antipodal case)**.** *Let X be an antipodal k-distance set on S^{d-1}. We define the polynomial $F_X(t)$ of degree $k-1$:*

$$F_X(t) := \prod_{\alpha \in A_{\mathrm{inn}}(X)\setminus\{-1\}} (t-\alpha) = \sum_{i=0}^{k-1} f_i G_i^{(d)}(t),$$

where the f_i are real and $f_i = 0$ for $i \equiv k \mod 2$. Then,

$$|X| \leq 2 \sum_{i \text{ with } f_i>0} h_i. \tag{16}$$

Corollary 3.4. *Let X be a two-distance set and $A_{\mathrm{inn}}(X) = \{\alpha, \beta\}$. Then, $F_X(t) := (t-\alpha)(t-\beta) = \sum_{i=0}^{2} f_i G_i^{(d)}(t)$ where $f_0 = \alpha\beta + 1/d$, $f_1 = -(\alpha+\beta)/d$ and $f_2 = 2/(d(d+2))$. If $\alpha + \beta \geq 0$, then*

$$|X| \leq h_0 + h_2 = \binom{d+1}{2}.$$

Musin proved this corollary by using a polynomial method in [21]. This corollary is used in proof of Theorem 4.13 in this paper. The following examples attain this upper bound in Corollary 3.4.

Example 3.5. Let U_d be a d-dimensional regular simplex. We define

$$X := \left\{ \frac{x+y}{2} \,\middle|\, x, y \in U_d, x \neq y \right\}$$

for $d \geq 7$. Then, X is a two-distance set on S^{d-1}, $|X| = d(d+1)/2$, $f_0 > 0$, $f_1 \leq 0$ and $f_2 > 0$.

Let us introduce some examples which attain the upper bounds in Theorem 3.2 and 3.3.

Corollary 3.6. *Let X be a one-distance set and $A_{\mathrm{inn}}(X) = \{\alpha\}$. Then, $F_X(t) := t - \alpha = \sum_{i=0}^{1} f_i G_i^{(d)}(t)$ where $f_1 = 1/d$ and $f_0 = -\alpha$. If $\alpha \geq 0$, then*

$$|X| \leq h_1 = d.$$

Clearly, a d-point $(d-1)$-dimensional regular simplex with a nonnegative inner product on S^{d-1} attains this upper bound.

Corollary 3.7. *Let X be an k-distance set on S^{d-1}. We have the Gegenbauer expansion $F_X(t) = \prod_{\alpha \in A_{\mathrm{inn}}(X)}(t-\alpha) = \sum_{i=0}^{k} f_i G_i^{(d)}(t)$. If $f_i > 0$ for all $i \equiv k \mod 2$ and $f_i \leq 0$ for all $i \equiv k-1 \mod 2$, then*

$$|X| \leq \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} h_{k-2i} = \binom{d+k-1}{k}.$$

The following examples attain their upper bounds.

Example 3.8. Let X be a tight spherical $(2k-1)$-design, that is, X is an antipodal k-distance set with $N'_d(k)$ points. There exist a subset Y such that $X = Y \cup (-Y)$ and $|X| = 2|Y|$. Y is an $(k-1)$-distance set with $\binom{d+k-2}{k-1}$ points. Defining $F_Y(t) := \sum_{i=0}^{k-1} f_i G_i^{(d)}(t)$, we have $f_i = 0$ for all $i \equiv k \mod 2$, and $f_i > 0$ for all $i \equiv k-1 \mod 2$.

4 Locally two-distance sets

In this section, we will consider locally two-distance sets. Recall that a locally two-distance set is said to be *proper* if it is not a two-distance set. The following examples imply that there are infinitely many proper locally two-distance sets when their cardinalities are small for their dimensions.

Example 4.1. Let U_d be the vertex set of a regular simplex in $\mathbb{R}^d$ and O be the center of the regular simplex. Let y be a point on the line passing through $x \in U_d$ and O. Then $U_d \cup \{y\}$ is a locally two-distance set. Except for finitely many exceptions, such locally two-distance sets are proper.

Example 4.2. Let $\{e_1, e_2, \ldots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Let

$$X = \{x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}\}$$

where

$$x_1 = e_1, \quad y_1 = -e_1$$

and

$$jx_j = e_{2j-2} + \sqrt{j^2-1}e_{2j-1}, \quad jy_j = e_{2j-2} - \sqrt{j^2-1}e_{2j-1}$$

for $2 \leq j \leq k-1$. Then X is a locally two-distance set and a k-distance set in $\mathbb{R}^{2k-3}$.

4.1 An upper bound for the cardinalities of locally two-distance sets

Lemma 4.3. (i) *Let $X \subset \mathbb{R}^d$ be a locally two-distance set with at least $d+2$ points. If $d \geq 2$, then there exist points $x, x' \in X$ $(x \neq x')$ such that $A(x) = A(x') = \{\alpha, \alpha'\}$ for some $\alpha, \alpha' \in \mathbb{R}_{>0}$ $(\alpha \neq \alpha')$.*
(ii) *Let X be a locally two-distance set in $\mathbb{R}^d$ with $n \geq d+2$ points. Then there exists $Y \subset X$ with $|Y| = n-d$ and $|A(x)| = 2$ for any $x \in Y$.*

Proof. (i) Let X be a locally two-distance set in $\mathbb{R}^d$ with more than $d+1$ points. Let $B(\alpha; x) = \{y \in X | d(x,y) = \alpha\}$ for any $x \in X$ and $\alpha \in A(x)$. Since $DS_d(1) = d+1$, there exists $x \in X$ such that $|A(x)| = 2$. Let $A(x) = \{\alpha_1, \alpha_2\}$, $Y_1 = B(\alpha_1; x)$ and $Y_2 = B(\alpha_2; x)$. For $y_1 \in Y_1$ and $y_2 \in Y_2$, if $d(y_1, y_2) \in \{\alpha_1, \alpha_2\}$, then we have $A(x) = A(y_1)$ or $A(x) = A(y_2)$ and this lemma holds. Otherwise, there exists $\beta \notin \{\alpha_1, \alpha_2\}$ such that $d(y_1, y_2) = \beta$ for all $y_1 \in Y_1$ and $y_2 \in Y_2$. Thus $A(y_i) = \{\alpha_i, \beta\}$ for any $y_i \in Y_i$ $(i = 1, 2)$. Moreover, $|Y_1| \geq 2$ or $|Y_2| \geq 2$ since $|X| \geq 4$.
(ii) Let X be a locally two-distance set in $\mathbb{R}^d$ with $n \geq d+2$ points. Let Y' be the set of all points in X with $|A(x)| = 1$. Then clearly $A(x) = A(x')$ for any $x, x' \in Y'$. Therefore Y' is a one-distance set and $|Y'| \leq d+1$. Moreover if $|Y'| = d+1$, then $Y' \cup \{y\}$ must be a one-distance set for any $y \in X \setminus Y'$, which is a contradiction. Thus $|Y'| \leq d$ and $|X \setminus Y'| \geq n-d$. $\square$

Remark 4.4. When we consider optimal locally two-distance sets, the condition $|X| \geq d+2$ in Lemma 4.3 is not so important because there is a lower bound $d(d+1)/2 \leq DS_d(2) \leq LDS_d(2)$ (cf. Example 3.5).

Let X be a locally two-distance set. A subset $Y \subset X$ is called a *saturated subset* if $|Y| \geq 2$ and Y is a maximal subset such that there exists α, β ($\alpha \neq \beta$) with $A_X(y) = \{\alpha, \beta\}$ for any $y \in Y$. Lemma 4.3 assures us that every locally two-distance set in $\mathbb{R}^d$ with at least $d+2$ points contains a saturated subset. Let $Y = \{y_1, y_2, \dots y_m\} \subset X$ be a saturated subset. Then Y is a two-distance set and $X \setminus Y$ is a locally two-distance set in the space $\{x \in \mathbb{R}^d | d(y_1, x) = d(y_2, x) = \cdots = d(y_m, x)\}$ by maximality. If $X \setminus Y \neq \emptyset$, then all points in Y are on a common sphere. Moreover $Y \cup \{x\}$ is a two-distance set for any $x \in X \setminus Y$.

Lemma 4.5. *Let $Y = \{y_0, y_1, \dots, y_{m-1}\} \subset \mathbb{R}^d$. Without loss of generality, we may assume that y_0 is the origin of $\mathbb{R}^d$. Let $\dim(Y)$ be the dimension of the space spanned by Y and $\mathrm{Sol}(Y) = \{x \in \mathbb{R}^d | d(y_0, x) = d(y_1, x) = \cdots = d(y_{m-1}, x)\}$. Then $\mathrm{Sol}(Y)$ is contained in a $(d - \dim(Y))$-dimensional affine subspace if $\mathrm{Sol}(Y) \neq \emptyset$.*

Proof. Let $y_i = (y_{i1}, y_{i2}, \dots, y_{id})$ for $1 \leq i \leq m-1$ and let $x = (x_1, x_2, \dots, x_d)$. For $1 \leq i \leq m-1$, $d(y_i, x) = d(y_0, x)$ implies

$$\sum_{k=1}^{d} y_{i\,k} x_k = \frac{1}{2} \sum_{k=1}^{d} {y_{i\,k}}^2.$$

Therefore

$$\mathrm{Sol}(Y) = \left\{ x \in \mathbb{R}^d | \begin{pmatrix} y_{1\,1} & y_{1\,2} & \cdots & y_{1\,d} \\ y_{2\,1} & y_{2\,2} & \cdots & y_{2\,d} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m-1\,1} & y_{m-1\,2} & \cdots & y_{m-1\,d} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix} \right\}$$

where

$$c_i = \frac{1}{2} \sum_{k=1}^{d} {y_{i\,k}}^2.$$

Since the rank of the above matrix is $\dim(Y)$, $\mathrm{Sol}(Y)$ is contained in a $(d - \dim(Y))$-dimensional subspace if $\mathrm{Sol}(Y) \neq \emptyset$. □

By Lemma 4.5, the following lemma holds.

Lemma 4.6. *Let X be a locally two-distance set in $\mathbb{R}^d$. Let $Y \subset X$ be a saturated subset and $\dim(Y) = i$. Then $X \setminus Y$ is a locally two-distance set with $\dim(X \setminus Y) \leq d - i$.*

Remark 4.7. Let X be a locally two-distance set and Y be a saturated subset of X in $\mathbb{R}^d$. Then we have $\dim(Y) \neq 0$ by Lemma 4.3. Moreover, if $\dim(Y) = d$, then $\dim(X \setminus Y) = 0$ by Lemma 4.6. In this case, $|X \setminus Y| \leq 1$ and X is a two-distance set. Therefore $1 \leq \dim(Y) \leq d-1$ for every saturated subset Y of a proper locally two-distance set X in $\mathbb{R}^d$. Moreover all points in Y are on a common sphere since $X \setminus Y \neq \emptyset$.

From the above remark, we have an upper bound for the cardinality of a proper locally two-distance set.

Theorem 4.8. *Let X be a proper locally two-distance set in $\mathbb{R}^d$. Then*

$$|X| \leq f(d)$$

where

$$f(d) = \max_{1 \leq i \leq d-1} \{DS_i^*(2) + LDS_{d-i}(2)\}.$$

In particular,

$$LDS_d(2) \leq \max\{DS_d(2), f(d)\}$$

Proof. Let X be a proper locally two-distance set in $\mathbb{R}^d$ and Y be a saturated subset of X and $i = \dim(Y)$. Then $1 \leq i \leq d-1$ and all points in Y are on a common sphere by Remark 4.7, so $|Y| \leq DS_i^*(2)$. On the other hand, $|X \setminus Y| \leq LDS_{d-i}(2)$ by Lemma 4.6. Therefore $|X| \leq DS_i^*(2) + LDS_{d-i}(2) \leq f(d)$. □

Corollary 4.9. *Every locally two-distance set in $\mathbb{R}^d$ with at least $d(d+1)/2+3$ points is a two-distance set. In particular $LDS_d(2) \leq \binom{d+2}{2}$.*

Proof. Let X be a proper locally two-distance set in $\mathbb{R}^d$. As we will see in Proposition 4.16, $LDS_d(2) \leq \binom{d+2}{2}$ for small d. Assume $LDS_i(2) \leq \binom{i+2}{2}$ for any $i \leq d-1$. By Theorem 4.8,

$$\begin{aligned}
|X| &\leq \max_{1\leq i\leq d-1}\{DS_i^*(2) + LDS_{d-i}(2)\} \\
&\leq \max_{1\leq i\leq d-1}\left\{\frac{i^2+3i}{2} + \frac{(d-i+2)(d-i+1)}{2}\right\} \\
&= \frac{1}{2}\max_{1\leq i\leq d-1}\{2i^2 - 2di + d^2 + 3d + 2\} \\
&= \frac{d(d+1)}{2} + 2
\end{aligned}$$

Therefore this corollary holds. □

Remark 4.10. (i) Since the set of midpoints of a regular simplex in $\mathbb{R}^d$ is a two-distance set with $d(d+1)/2$ points, Corollary 4.9 implies $DS_d(2) \leq LDS_d(2) \leq DS_d(2) + 2$. For $d \leq 8$, $d \neq 3$, we will see that $DS_d(2) = LDS_d(2)$ in Proposition 4.16.
(ii) For spherical cases, similarly we have $DS_d^*(2) \leq LDS_d^*(2) \leq DS_d^*(2) + 1$.

Problem 4.11. When does $DS_d(2) < LDS_d(2)$ (resp. $DS_d^*(2) < LDS_d^*(2)$) hold?

We will give partial results for general cases in Section 4.2 and give an answer for $d \leq 8$ in Section 4.4.

4.2 Partial answer to Problem 4.11

Lemma 4.12. (i) *Let X be a proper locally two-distance set in $\mathbb{R}^d$ for $d \geq 3$. If $d(d+1)/2 < |X|$, then there exist $N_{d-1}(2)$-point two-distance set in S^{d-2} or $(N_{d-1}(2)-1)$-point two-distance set Y in S^{d-2} with $A(Y) = \{1, 2/\sqrt{3}\}$.*
(ii) *Let X be a proper locally two-distance set in S^{d-1} for $d \geq 3$. If $d(d+1)/2 < |X|$, then there exist $N_{d-1}(2)$-point two-distance set Y in S^{d-2} with $\sqrt{2} \in A(Y)$ or $A(Y) = \{\alpha, \alpha/\sqrt{\alpha^2-1}\}$.*

Proof. (i) For the case where $d \in \{3,4\}$, we will prove this proposition directly in Proposition 4.16. Therefore we assume that $d \geq 5$ in this proof. Let X be a proper locally two-distance set in $\mathbb{R}^d$ with more than $d(d+1)/2$ points and let Y be a saturated subset of X. We may assume that Y has maximum cardinality among saturated subsets of X. Let $i = \dim(Y)$. Then $1 \leq i \leq d-1$ since Y is a saturated subset and X is not a two-distance set. If $2 \leq i \leq d-2$, then $d(d+1)/2 \geq |X|$ for $d \geq 5$ by Theorem 4.8. Moreover if $i = 1$, then $|Y| \leq 2$ and $|X \setminus Y| \geq d(d+1)/2 - 2 > d(d-1)+3$ for $d \geq 3$. Since $X \setminus Y$ is a locally two-distance set in $\mathbb{R}^{d-1}$, $X \setminus Y$ is a two-distance set by Corollary 4.9. By Lemma 4.3, $X \setminus Y$ contains a saturated subset Y' and $|Y'| > |Y|$. This is a contradiction to the assumption. Therefore $i = d-1$. Since $|X| \geq d(d+1)/2+1 = N_{d-1}(2)+2$ and $|X \setminus Y| \leq LDS_1(2) = 3$, $|Y| \geq N_{d-1}(2)-1$. It is enough to consider the case $|Y| = N_{d-1}(2)-1$, otherwise $|Y| = N_{d-1}(2)$ and this proposition holds. In this case, $|X \setminus Y| = 3$. Let $A(Y) = \{\alpha, \beta\}$ and $X \setminus Y = \{x_1, x_2, x_3\}$. For any $i \in \{1,2,3\}$, $A(x_i) \neq \{\alpha, \beta\}$ since Y is a saturated subset. Moreover $d(x_i, y) = \alpha$ for all $y \in Y$ or $d(x_i, y) = \beta$ for all $y \in Y$. Since $\dim(X \setminus Y) = 1$, there are four possibilities for the x_i. Without loss of generality, we may assume $d(x_1, y) = d(x_2, y) = \alpha$ for all $y \in Y$ and $d(x_3, y) = \beta$ for all $y \in Y$. Then $d(x_1, x_3) = d(x_2, x_3) = \gamma$ for $\gamma \notin \{\alpha, \beta\}$ and $d(x_1, x_2) = \alpha$. It follows from these conditions that Y is an $(N_{d-1}(2)-1)$-point two-distance set Y in S^{d-2} with $A(Y) = \{1, 2/\sqrt{3}\}$.

(ii) Let X be a proper locally two-distance set in S^{d-1} with more than $d(d+1)/2$ points and let Y be a saturated subset of X. Similar to the above case, we may assume $i = \dim(Y) = d-1$. Since $|X| \geq N_{d-1}(2)+2$ and $|X \setminus Y| \leq LDS_1^*(2) = 2$, $|Y| \geq N_{d-1}(2)$. Therefore, $|Y| = N_{d-1}(2)$. $\square$

Theorem 4.13. (i) *If there exists a proper locally two-distance set X in $\mathbb{R}^d$ with more than $d(d+1)/2$ points, then there exists an $N_{d-1}(2)$-point two-distance set in S^{d-2}.*
(ii) *If there exists a proper locally two-distance set X in S^{d-1} with more than $d(d+1)/2$ points, then there exists an $N_{d-1}(2)$-point two-distance set in S^{d-2}. In particular, a locally two-distance set in S^{d-1} with more than $d(d+1)/2$ points is a subset of a tight spherical five-design.*

Proof. (i) Let X be a proper locally two-distance set in $\mathbb{R}^d$ with more than $d(d+1)/2$ points. We assume that X does not contain $N_{d-1}(2)$-point two-distance set in S^{d-2}. Then X contains $(N_{d-1}(2)-1)$-point two-distance set $Y \subset S^{d-2}$ with $A(Y) = \{1, 2/\sqrt{3}\}$ by Lemma 4.12(i) . However there does not exist such a two-distance set Y by Corollary 3.4.
(ii) This is clear by Lemma 4.12 (ii) and Remark 2.4. $\square$

Remark 4.14. Since $d(d+1)/2 \leq DS_d(2)$ (resp. $d(d+1)/2 \leq DS_d^*(2)$), the assumption in Theorem 4.13 (i) (resp. (ii)) can be replaced by $DS_d(2) < LDS_d(2)$ (resp. $DS_d^*(2) < LDS_d^*(2)$).

4.3 Classifications of optimal two-distance sets

Euclidean cases $DS_d(2)$ is determined for $d \leq 8$ and optimal two-distance sets are classified for $d \leq 7$ (Kelly [18], Croft [9], Einhorn-Schoenberg [14] and Lisoněk [20]). We introduce the results in this subsection.

$d=2$: $DS_2(2)$ and the optimal planar two-distance set is isomorphic to the set of vertices of a regular pentagon (Kelly [18], Einhorn-Schoenberg [14]). We denote the set of vertices of the regular pentagon with side length 1 by R_5. Then $A(R_5) = \{1, \tau\}$ where $\tau = (1+\sqrt{5})/2$.

$d=3$: $DS_3(2)$ and there are exactly six optimal two distance sets in $\mathbb{R}^3$ (Croft [9], Einhorn-Schoenberg [14]). They are the set of vertices of a regular octahedron, a right prism which has a equilateral triangle base and square sides and the remaining four sets are subsets of a regular icosahedron.

$d=4$: $DS_4(2) = 10$ and the optimal two-distance set in $\mathbb{R}^4$ is isomorphic to the set of midpoints of the edges of a regular simplex in $\mathbb{R}^4$. This set corresponds to the Petersen graph.

$d=5$: $DS_5(2) = 16$ and the optimal two-distance set in $\mathbb{R}^5$ is isomorphic to the set given by the Clebsch graph. Points of the set are given by the following.

$$-e_i + \sum_{k=1}^{5} e_k \quad (1 \leq i \leq 5),$$

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$$e_i + e_j \quad (1 \leq i < j \leq 5)$$

and the origin O of $\mathbb{R}^5$.

$d=6$: $DS_6(2) = 27$ and the optimal two-distance set in $\mathbb{R}^6$ is isomorphic to the set obtained from the Schläfli graph.

$d=7$: $DS_7(2) = 29$ and the optimal two-distance set in $\mathbb{R}^7$ is isomorphic to the set which is given by the following points.

$$-e_i + \frac{1}{7}(3+\sqrt{2}) \sum_{k=1}^{7} e_k \quad (1 \leq i \leq 7),$$

$$e_i + e_j \quad (1 \leq i < j \leq 7)$$

and

$$\frac{1}{7}(2+3\sqrt{2})\sum_{k=1}^{7} e_k.$$

$d = 8$: A two-distance set in $\mathbb{R}^8$ with $\binom{10}{2} = 45$ points is known. Let

$$X_1 = \{e_i - \frac{1}{12}\sum_{k=1}^{8} e_k | i = 1, 2, \ldots 8\} \cup \{-\frac{1}{3}\sum_{k=1}^{8} e_k\}$$

and

$$X_2 = \{-(x+y) | x, y \in X_1, x \neq y\}$$

Then X_1 is the vertex set of a regular simplex and $X_1 \cup X_2$ is a two-distance set with $A(X_1 \cup X_2) = \{\sqrt{2}, 2\}$

Spherical cases For $2 \leq d \leq 6$, every optimal two-distance set in $\mathbb{R}^d$ is on a sphere. Optimal two-distance sets in S^6 are given from three Chang graphs or the set of midpoints of edges of a regular simplex in $\mathbb{R}^7$. Moreover, Musin [21] determined $DS^*_d(2)$ for $7 \leq d < 40$.

Theorem 4.15. *$DS^*_d(2) = d(d+1)/2$ for the cases where $7 \leq d \leq 21, 24 \leq d < 40$. When $d = 22, 23$, $DS^*_{22}(2) = 275$ and $DS^*_{23}(2) = 276$ or 277.*

4.4 Optimal locally two-distance sets

Euclidean cases By using classifications of optimal two-distance sets and Theorem 4.8, we have the following proposition.

Proposition 4.16. *Every optimal locally two-distance set in $\mathbb{R}^d$ is a two-distance set for $d = 2, 4, 5, 6, 8$. Moreover there are four seven-point locally two-distance set in $\mathbb{R}^3$ up to isomorphism and five 29-point locally two-distance set in $\mathbb{R}^7$ up to isomorphism. In particular $DS_d(2) = LDS_d(2)$ for $d = 1, 2, 4 \leq d \leq 8$ and $LDS_3(2) = 7$.*

Proof. $d = 1$: It is clear that every three-point set in $\mathbb{R}^1$ which is not a one-distance set is a locally two-distance set and that there is no four-point locally two-distance set in $\mathbb{R}^1$.

For $2 \leq d \leq 7$, we classify optimal locally two-distance sets in $\mathbb{R}^d$. For each case, we pick a saturated subset Y of X and we let $Y' = X \setminus Y$. Note that if X is not a two-distance set, then $1 \leq \dim(Y) \leq d-1$.

$d = 2$: We will classify five-point locally two-distance sets X in $\mathbb{R}^2$. We may assume that $\dim(Y) = 1$ and $|Y| = 2$, otherwise X is a two-distance set. Let $Y = \{y_1, y_2\}$, $Y' = \{x_1, x_2, x_3\}$ and $A(y_1) = A(y_2) = \{\alpha, \beta\}$. Without of generality, we may assume $d(x_1, y_i) = d(x_2, y_i) = \alpha$ and $d(x_3, y_i) = \beta$ for $i \in \{1, 2\}$ since there are exactly four possibilities for the x_j. If $d(x_1, x_3) \in \{\alpha, \beta\}$, then $A(x_1) = \{\alpha, \beta\}$ or $A(x_3) = \{\alpha, \beta\}$. This is a contradiction to the maximality of the saturated subset Y. So $d(x_1, x_3) = \gamma \notin \{\alpha, \beta\}$. Similarly $d(x_2, x_3) = \gamma$. Therefore x_3 is a midpoint of both the segment y_1y_2 and the segment x_1x_2. It is easy to check that such a locally two-distance set does not exist. Therefore $\dim(Y) \neq 1$ and X is a two-distance set. By the classification of five-point two-distance sets in $\mathbb{R}^2$, $X = R_5$.

$d = 3$: We will classify seven-point locally two-distance sets X in $\mathbb{R}^3$. We may assume $1 \leq \dim(Y) \leq 2$, otherwise X is a two-distance set. We need to consider two cases (a) $\dim(Y) = 1$ and (b) $\dim(Y) = 2$.
(a) In this case, $|Y| = 2$ and $Y' = R_5$ by the above classification. Let $Y = \{y_1, y_2\}$ and $Y' = \{x_1, x_2, \ldots, x_5\}$. Then $d(x_j, y_i) = 1$ for any $j \in \{1, 2\}$ and $i \in \{1, 2, \ldots, 5\}$ or $d(x_j, y_i) = \tau$ for any $j \in \{1, 2\}$ and $i \in \{1, 2, \ldots, 5\}$. In this case, there are two seven-point locally two-distance sets up to isomorphism.
(b) In this case, $|Y| \in \{4, 5\}$. If $|Y| = 4$, then $|Y'| = 3$. Similar to the case where $d = 2$, there exists a point $x \in Y'$ which is the midpoint of the other two points. Then $Y \cup \{x\}$ is a five-point locally two-distance set in $\mathbb{R}^2$ and x is a center of the circle passing through other four points. By the classification of

five-point locally two-distance sets in $\mathbb{R}^2$, such a locally two-distance set does not exist. If $|Y| = 5$, then $|Y'| = 2$. In this case, $Y = R_5$ and there are four locally two-distance sets up to isomorphism. These sets contains the sets in case (a).

$d = 4$: We will classify ten-point locally two-distance sets X in $\mathbb{R}^4$. If $\dim(Y) \neq 2$, then X is a two-distance set or $|X| < 10$. Therefore we assume $\dim(Y) = 2$. Then $|Y| = |Y'| = 5$ and both Y and Y' are sets of vertices of a regular pentagon. Let

$$Y = \{(\cos \frac{2\pi j}{5}, \sin \frac{2\pi j}{5}, 0, 0) | j = 0, 1, \dots 4\}$$

and

$$Y' = \{(0, 0, r \cos \frac{2\pi j}{5}, r \sin \frac{2\pi j}{5}) | j = 0, 1, \dots 4\}.$$

Then $d(x, y) = \sqrt{1 + r^2} > 1$ for any $y \in Y$ and $x \in Y'$. Therefore we may assume $d(x, y) = \tau$ where $\tau = (1 + \sqrt{5})/2$. Then $r = \sqrt{\tau}$ and $A(x) = \{\tau^{1/2}, \tau, \tau^{3/2}\}$ for $x \in Y'$. This is not a locally two-distance set. Therefore a ten-point locally two-distance set is a two-distance set.

$d = 5$: We will classify sixteen-point locally two-distance sets X in $\mathbb{R}^5$. Since $DS_i^*(2) + LDS_{d-i}(2) < 16$ for $1 \leq i \leq 4$, X is a two-distance set.

$d = 6$: We will classify 27-point locally two-distance sets X in $\mathbb{R}^6$. By Corollary 4.9, every 27-point locally two-distance set in $\mathbb{R}^6$ is a two-distance set.

$d = 7$: We will classify 29-point locally two-distance sets X in $\mathbb{R}^7$. If $\dim(Y) \notin \{1, 6\}$, then X is a two-distance set or $|X| < 29$. We divide into two cases: (a) $\dim(Y) = 1$ and (b) $\dim(Y) = 6$.
(a) In this case, similar to the classification of case (a) for $d = 3$, we prove that there are two 29-point locally two-distance sets up to isomorphism.
(b) In this case, similar to the classification of case (b) for $d = 3$, we can prove that there are four locally two-distance sets which contain the sets in case (a).

$d = 8$: We will consider 45-point locally two-distance sets in $\mathbb{R}^8$. By Corollary 4.9, every 45-point locally two-distance set in $\mathbb{R}^8$ is a two-distance set. □

Spherical cases For spherical cases, we have the following proposition by Theorem 4.13 and Theorem 4.15.

Proposition 4.17. *$LDS_d^*(2) = DS_d^*(2)$ for $2 \leq d < 40$ and $d \notin \{3, 7, 23\}$. When $d \in \{3, 7, 23\}$, $LDS_3^*(2) = 7$, $LDS_7^*(2) = 29$ and $LDS_{23}^*(2) = 277$. In particular, there is a unique optimal locally two-distance set in S^{d-1} if $d \in \{3, 7\}$ and there is a unique optimal locally two-distance set in S^{23} if $DS_{23}^*(2) = 276$.*

4.5 Optimal locally three-distance sets

It seems difficult to determine $LDS_d(k)$ and classify the optimal configurations for $k \geq 3$. However there is a result for $k = 3$ and $d = 2$ by Erdős-Fishburn [16] and Fishburn [17].

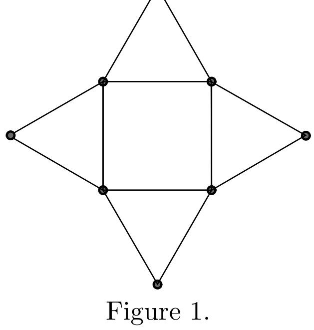

Figure 1.

Proposition 4.18. (i) *Let X be an eight-point planar set. Then* $\sum_{P\in X} |A_X(P)| \geq 24$.
(ii) *Every eight-point planar set X with* $\sum_{P\in X} |A_X(P)| = 24$ *is similar to Figure 1.*
(iii) *Every eight-point locally three-distance set in $\mathbb{R}^2$ is similar to Figure 1. In particular,* $LDS_3(3) = 8$.

Proof. (i), (ii) See [16], [17].
(iii) This is immediate from (i), (ii). □

The second author proved that $DS_3(3) = 12$ and that every twelve-point three-distance set in $\mathbb{R}^3$ is similar to the set of vertices of a regular icosahedron ([24]).

Problem 4.19. Is every locally three-distance set in $\mathbb{R}^3$ with twelve points similar to the set of vertices of a regular icosahedron?

In fact, there are many differences between k-distance sets and locally k-distance sets when cardinalities are small. Moreover we saw that $DS_d(k) < LDS_d(k)$ for some cases. However no known optimal k-distance sets are locally $(k-1)$-distance sets.

Problem 4.20. Are there any optimal k-distance sets which are locally $(k-1)$-distance sets?

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